An Empirical Research on Identifiability and Q-matrix Design for DINA model

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ABSTRACT

In most contexts of student skills assessment, whether the test material is administered by the teacher or within a learning environment, there is a strong incentive to minimize the number of questions or exercises administered in order to get an accurate assessment. This minimization objective can be framed as a Q-matrix design problem: given a set of skills to assess and a fixed number of question items, determine the optimal set of items, out of a potentially large pool, that will yield the most accurate assessment. In recent years, the Q-matrix identifiability under DINA/DINO models has been proposed as a guiding principle for that purpose. We empirically investigate the extent to which identifiability can serve that purpose. Identifiability of Q-matrices is studied throughout a range of conditions in an effort to measure and understand its relation to student skills assessment. The investigation relies on simulation studies of skills assessment with synthetic data. Results show that identifiability is an important factor that determines the capacity of a Q-matrix to lead to accurate skills assessment with the least number of questions.

1. INTRODUCTION

Consider a set of items intended to assess a student's mastery over a set of skills, or knowledge components (KC). These items, along with the set of skills, can be designed to test a single skill at once. Or, they can be designed to involve two or more skills. A test composed of a fixed number of items can either be composed of a mixture of single and multiple skills items, or composed of one type of items only. Skills can themselves be defined so as to facilitate the creation of task/problem items that involve single skill per item, or multiple skills per items. By which principles should a teacher choose among these different options?

This paper addresses this question, with the general objective of designing a test that will bring the most accurate assessment of a student's skill mastery state with the least number of questions items.

The investigation is framed within the DINA model, which was a widely researched model and originally proposed in the research of a rule space method for obtaining diagnostic scores (Tatsuoka, 1983). In this model, question items can involve one or more skills, and all skills are required in order to succeed the question, while a success can still occur through a guessing factor, and failure can also occur through a slip factor.

2. Q-MATRIX, DINA MODEL AND IDENTIFIABILITY

The mapping of items to skills is referred to as a Q-matrix, where items are mapped to latent skills whose mastery is deemed necessary in order for the student to succeed at the items. An item can represent a question, an exercise, or any task that can have a positive or negative outcome. In the DINA model, the conjunctive version of the Q-matrix is adopted: all skills are considered necessary for success.

In the last decade, a number of papers have been devoted to deriving a Q-matrix from student test results data (Barnes, 2010; Liu, Xu, & Ying, 2012; Desmarais, Xu, & Beheshti, 2015; P. Xu & Desmarais, 2016). Another line of research on Q-matrices has been devoted to refine or to validate an expert-given Q-matrix (de la Torre & Chiu, 2015; Chiu, 2013; Desmarais & Naceur, 2013). While the problems of deriving or refining a Q-matrix from data are related to Q-matrix design, they do not provide insight into how best to design them.

In parallel to these investigations, some researchers have looked at the question of the identifiability. The general idea behind identifiability is that two or more configurations of model parameters can be considered as equivalent. Sets of parameters will be considered equivalent if, for example, their likelihood is equal given a data sample. Or, conversely, if the parameters are part of a generative model, two sets of equivalent parameters would generate data having the same characteristics of interest, in particular equal joint probability distributions (see Doroudi & Brunskill, 2017, for more details).

The issue of identifiability for student skills assessment was first researched in multiple diagnosis model comparison (Yan, Almond, & Mislevy, 2004), Bayesian Knowledge Tracing (Beck & Chang, 2007) and later discussed by more researchers (van De Sande, 2013; Doroudi & Brunskill, 2017). A mathematically rigorous treatment Q-matrix identifiability under the DINA/DINO setting was presented under zero slip and guess parameters (Chiu, Douglas, & Li, 2009), and under known slip and guess (Liu, Xu, & Ying, 2013), and finally under unknown slip and guess parameters (Chen, Liu, Xu, & Ying, 2015). An overall discussion can also be found (G. Xu & Zhang, 2015; Qin et al., 2015). These studies provide theoretical basis to derive Q-matrices from data, but not to the design of Q-matrices itself. In this paper, we consider the identifiability of the Q-matrix with

regards to the DINA model.

Identifiability is a general concept for statistical models. Its formal definition is:

Definition (1) (Casella & Berger, 2002) A parameter θ for a family of distribution $f(x|\theta : \theta \in \Theta)$ is *identifiable* if distinct values of θ correspond to distinct pdfs or pmfs. That is, if $\theta \neq \theta'$, then $f(x|\theta)$ is not the same function of x as $f(x|\theta')$.

The DINA model has parameters $\theta = \{Q, p, s, g\}$, where Q is the Q-matrix. p is the categorical distribution parameter for all student profile categories. That is, it indicates the probability that a student belongs to each profile category. For example, in a 3-skill case, there are $2^3 = 8$ categories for students to belong to, and the 8-component probability vector of students belongs to each of these categories is the model parameter p. Finally, s and g are both vectors denoting the slip and guess of each item.

The identifiability of all parameters in DINA model have been thoroughly investigated and several theorems are given (G. Xu & Zhang, 2015). But for the Q-matrix design problem that is the focus of this paper, we solely need to ensure that the model parameter p is identifiable, meaning that we can distinguish different profile categories. Fortunately, for the case when s and g are known, the requirement is easily satisfied, since it only requires the Q-matrix to be *complete*.

Definition (2) (Chen et al., 2015) The matrix Q is *complete* if $\{e_i : i = 1, ..., K\} \subset R_Q$, where K is the number of skills (columns of Q), R_Q is the set of row vectors of Q, and e_i is a row vector such that the i-th element is one and the rest are zero (i.e. a binary unit vector, also known as a "one-hot vector"). Stated differently, the rows of the identity matrix, $I_{K \times K}$, must be in Q for this matrix to be complete.

And the heart of the current investigation is based on the following proposition:

Proposition (Chen et al., 2015) Under the DINA and DINO models, with Q, s and g being known, the population proportional parameter p is *identifiable* if and only if Q is *complete*.

We show an example of Q-matrix that is not complete below for better illustration.

$$
\begin{array}{c}
 \\
q_1 \\
q_2 \\
q_3
\end{array}
\begin{array}{ccc}
k_1 & k_2 & k_3 \\
\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{array}\right]
\end{array}
$$

This Q-matrix does not contain $e_2 : [0, 1, 0]$ or $e_3 : [0, 0, 1]$, and is therefore not complete, even though its items (rows) cover all skills (columns). Using this Q-matrix under DINA model setting entails that the model parameters are not identifiable according to the proposition above, and would in turn compromise student profile diagnosis. In fact, students who only master skill 2 and students who only master skill 3 are indistinguishable under this Q-matrix.

But while the use of a non identifiable Q-matrix should be avoided according to the proposition, the question remains:

among all the complete Q-matrix, which ones are most efficient for student profile diagnosis?

In the next section, we investigate empirically the Q-matrix design options in light of the *completeness* requirement, using synthetic student performance data with the DINA model. Synthetic data is essential for this investigation because we need to know the underlying ground truth. We return to the issue of using real data in the conclusion.

3. EXPERIMENT

The Q-matrix design problem is essentially an optimization problem. Basically, we have a pool of Q-matrices, and each of them is formed by a selection with replacement from a pool of q-vectors. Each Q-matrix will yield some capacity to diagnose students, as measured by a loss function. We aim to choose a Q-matrix that minimizes the loss function.

Our experiments follow a Bayesian framework to diagnose students under DINA Q-matrices. First, we use one-hot encoding to denote all profile categories. Set M to be the number of profile categories. Then, in the 3-skill case, the $M = 8$ profile categories pc_i are:

$$
\begin{array}{c}
 \\
pc_1 \\
pc_2 \\
pc_3 \\
pc_4 \\
pc_5 \\
pc_6 \\
pc_7 \\
pc_8
\end{array}
\begin{array}{ccc}
k_1 & k_2 & k_3 \\
\left[\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{array}\right]
\end{array}
$$

Therefore, a student belonging to profile pc_1 is encoded as a binary unit vector $\alpha_1 = (1, 0, 0, 0, 0, 0, 0, 0)$, and so on for pc_2 encoded as $\alpha_2 = (0, 1, 0, 0, 0, 0, 0, 0)$, ..., and pc_8 encoded as $\alpha_8 = (0, 0, 0, 0, 0, 0, 0, 1)$. The DINA model parameter p is represented as a probability vector $p = (p_1, p_2, ..., p_8) = (P(\alpha_1), P(\alpha_2), ..., P(\alpha_8))$. Then, we set the prior of each student profile to be:

$$\alpha_0 = (1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8)$$

With the conditional independence assumed (i.e, conditioned on a given profile category, the probability to answer each question correct is independent), the likelihood is given by (De La Torre, 2009; Chen et al., 2015):

$$
\begin{aligned}
L(p, Q, s, g|X) &= P(X|p, Q, s, g) \\
&= \prod_{i=1}^{I} \sum_{\alpha} p_\alpha P(X_i|\alpha, Q, s, g) \\
&= \prod_{i=1}^{I} \sum_{\alpha} p_\alpha \prod_{j=1}^{J} P_j(\alpha)^{X_{ij}} [1 - P_j(\alpha)]^{1-X_{ij}}
\end{aligned}
$$

(1)

in which X is the response matrix and X_i is the i-th row, I is the number of records (students), J is the number of questions. $P_j(\alpha)$ is the probability of student profile α to answer correctly of question j, notice α in 3-skill case has only 8 possible values, for any of them $\alpha_m, m = 1, ..., 8$, the probability is given by DINA model

$$P_j(\alpha_m) = P(X_{ij} = 1|\alpha_m) = g_j^{1-\eta_{mj}} (1 - s_j)^{\eta_{mj}}$$

in which η_{mj} is the latent response of profile α_m to question j, that is, the response when slip and guess is 0. It can be calculated by

$$\eta_{mj} = \prod_{k=1}^{K} \alpha_{mk}^{q_{jk}}$$

where K is the number of skills and q_{jk} is the (j,k)-th element of Q-matrix Q.

Given the prior and likelihood, the posterior $\hat{\alpha}$ for each student can be calculated. It has the form:

$$\hat{\alpha} = (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5, \hat{p}_6, \hat{p}_7, \hat{p}_8)$$

and we then calculate the loss between this posterior and the true profile α_{true}, which is one of the one-hot encoding vector.

For any Q-matrix configuration, the loss function is defined by

$$loss(Q) = \sum_{i \in \text{students}} \|\hat{\alpha}_i - \alpha_{\text{true}}\|^2$$

To implement the experiment, for each Q-matrix configuration, we generate a response matrix based on the DINA model given fixed slip and guess parameters, using function 'DINAsim' from the R package *DINA* (Culpepper, 2015). Then, we calculate the posterior estimation for all students and evaluate the total loss. The reported result is an average loss of 100 runs.

In our experiments, we consider the 3-skills and 4-skills cases. For the 3-skills case, experiments are conducted with $N = 200$ students, of which 25 students fall into each of 8 categories. For the 4-skills case, we use $N = 400$ students, of which 25 students fall into each of 16 categories.

3.1 Experiment 1: Comparison of three strategies
In the first experiment, we compare three different Q-matrix design strategies. They are all based on repetition of a specific pool of q-vectors.

- Strategy 1 (Q-matrix 1): Using the identifiability condition (definition (1)) by using only combinations of the vectors $\{e_i : i = 1, ..., K\}$ (binary unit vectors, or one-hot encodings).

- Strategy 2 (Q-matrix 2): Using the vectors $\{e_i : i = 1, ..., K\}$ plus an all-one vector $(1, 1, 1)$ (in 3-skill case) or $(1, 1, 1, 1)$ (in 4-skill case). This is inspired by orthogonal array design, which is a commonly seen design of experiments (Montgomery, 2017).

- Strategy 3 (Q-matrix 3): Repeatedly using all q-vectors.

For the 3-skills case, all these three Q-matrices are shown in Figure 1. The general pattern is to recycle the rows above the lines denoted by ...[..., ..., ...].

The 4-skills case is similar, which is omitted here. Results of these two cases are shown in Figure 2a and Figure 2b.

Figure 1: Q-matrix design strategies

3.2 Experiment 2: Find best configuration
The second experiment takes the brute force approach. We directly examine all possible Q-matrix configurations. First, for a given pool of q-vectors to choose from and an integer indicating the number of questions, we need to know the number of possible configurations of Q-matrices we have. This is equivalent to a classical combinatorial problem, that is, to allocate marbles (q-vectors) to bins (questions). It can be easily computed by combinatorial coefficients and interpreted by using stars and bars methods. For example, in 3-skills case, we have 7 q-vectors, and if we have 4 questions to allocate them, then we have $\binom{4+7-1}{7-1} = 210$ possible configurations. This number grows up sharply as a number of questions increases or number of patterns increases. As a comparison, in the 4-skills case, if we have 5 questions to allocate them, then we have $\binom{5+15-1}{15-1} = 11628$ possible configurations.

For each configuration, we calculate the MAP estimation for all categories of each student, and compare with the one-hot encoding for their true categories. The total loss is reported as the performance index.

Figure 3 shows the results of 6 combinations of different numbers of skills and questions:

- 3-skills case, 4 questions: Figure 3a, Figure 3b

- 3-skills case, 8 questions: Figure 3c, Figure 3d

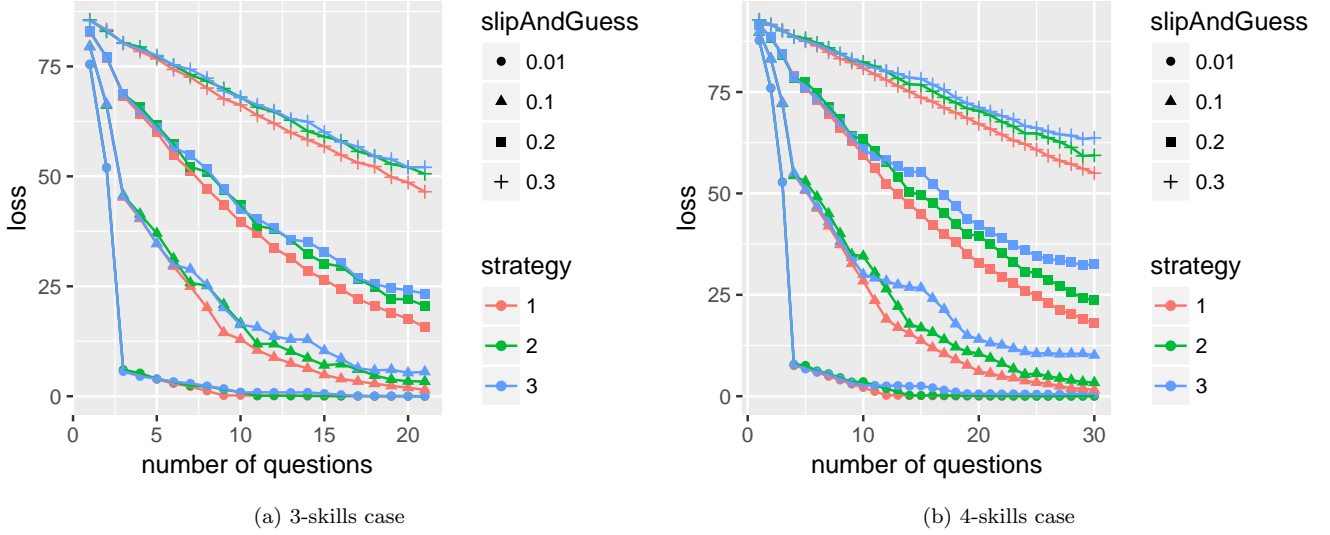- 4-skills case, 5 questions: Figure 3e, Figure 3f

(a) 3-skills case   (b) 4-skills case

Figure 2: Experiment 1: Three Strategy Comparison on 3- and 4-skills cases

4. DISCUSSION

From the result of experiment 1 we can see that strategy 1 always works better than the other two strategies, meaning that simply repeating the vectors $\{e_i : i = 1, ..., K\}$ in Q-matrix design, without using any combination of skills, yields better student diagnosis performance.

From the result of experiment 2, when slip and guess parameters are as low as 0.01, we can see obvious graded patterns among different configurations. This can be explained by the the distinguishability of a Q-matrix. For example, in Figure 3a, we can see there are 7 layers. In fact, the first layer consisted of Q-matrix that can only cluster students into 2 categories. One example of such a Q-matrix is

$$
\begin{array}{c}
 \\
q_1 \\
q_1 \\
q_1 \\
q_1
\end{array}
\begin{array}{ccc}
k_1 & k_2 & k_3 \\
\left[\begin{array}{ccc}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{array} \right]
\end{array}
$$

This Q-matrix can only discriminate between a student that mastered skill 1 or not. We know that there are in fact 8 categories of students, the 7 layers in Figure 3a from top to bottom correspond to the Q-matrix that can separate students into 2 to 8 categories. We can see that complete Q-matrices always fall in the bottom layer, which concurs with the proposition of Section 2. The 4-skills case is similar in Figure 3e.

When slip and guess parameter increase, the points become more divergent, as can be seen by comparison between figures 3a and 3b. In order to see some greater details, we distinguish three types of Q-matrices.

- Type I: Complete and confined, meaning it is only consisted of vectors $\{e_i : i = 1, ..., K\}$.

- Type II: Complete but not confined, meaning it not only contains all vectors $\{e_i : i = 1, ..., K\}$, but also

contains at least one other q-vector.

- Type III: Incomplete Q-matrix.

Type I and Type II Q-matrices performs the same when slip and guess are low (figures 3a, 3e), but when they get higher, Type I Q-matrices show a better performance (figures 3b, 3f).

However, when more questions are involved in a high slip and guess condition, the performance becomes more unstable. Therefore, we again consider more subtypes. In 3-skills case for 8 questions, we consider three subtypes below.

- Subtype 1: Q-matrix contains each component of $\{e_i : i = 1, ..., K\}$ at least twice.

- Subtype 2: Other situations (e.g A complete Q-matrix but all the other vectors are just repeated e_1).

- Subtype 3: Q-matrix contains all q-vectors.

From Figure 3d we can see that the subtype 1 (denoted by triangle) shows better performance than subtype 2, meaning that repeating the whole set of $\{e_i : i = 1, ..., K\}$ is a better strategy just like the strategy 1 we used in experiment 1. Subtype 3 corresponds to the strategy 3 in experiment 1, it has only 7 possible configurations in 8-question setting and we can see that they do not perform well.

Therefore, we argue that the best Q-matrix design is to use only the vectors $\{e_i : i = 1, ..., K\}$ since it offers quicker convergence speed (as shown in experiment 1) and better robustness against slip and guess (as shown both in experiments 1 and 2).
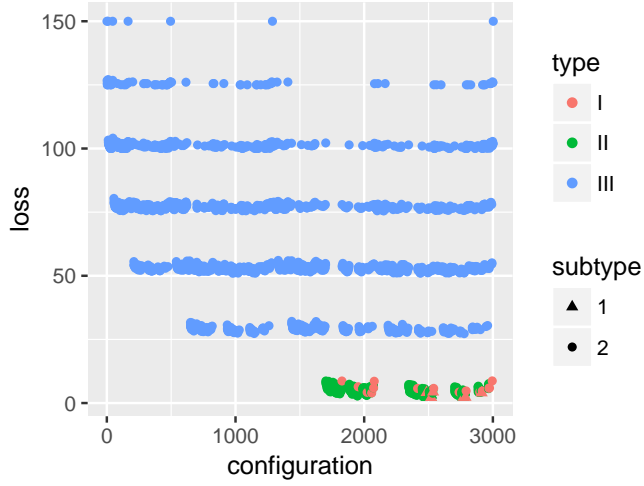
5. CONCLUSION

This work is still in an early stage and has limitations, in particular because it is conducted with synthetic data. But
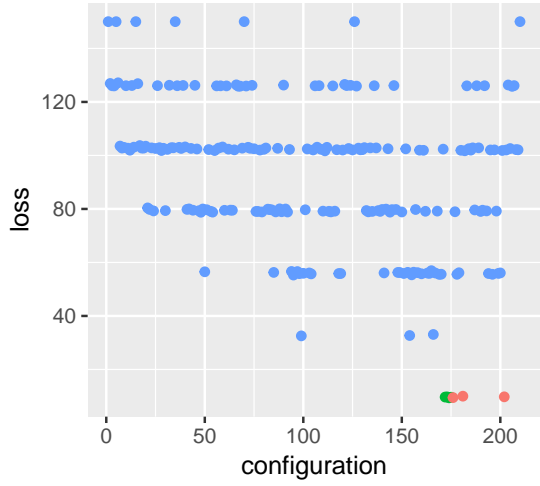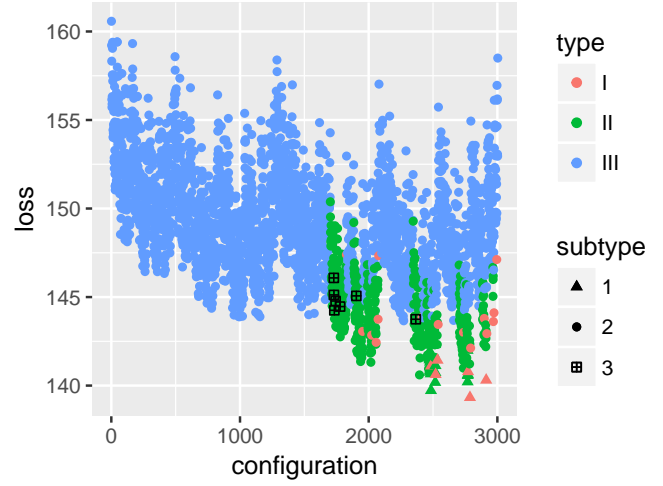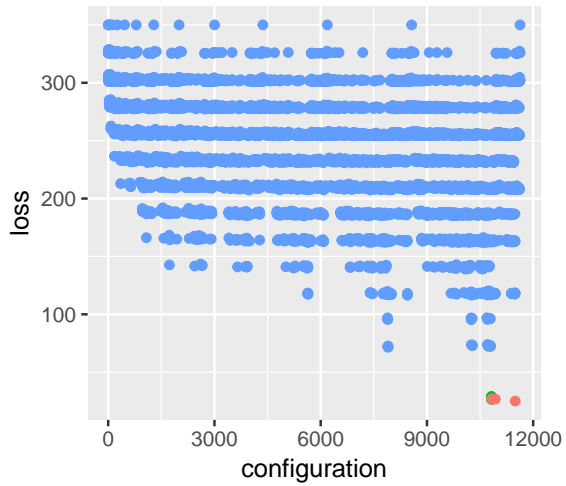
(a) 3-skills case, slip=guess=0.01, J=4
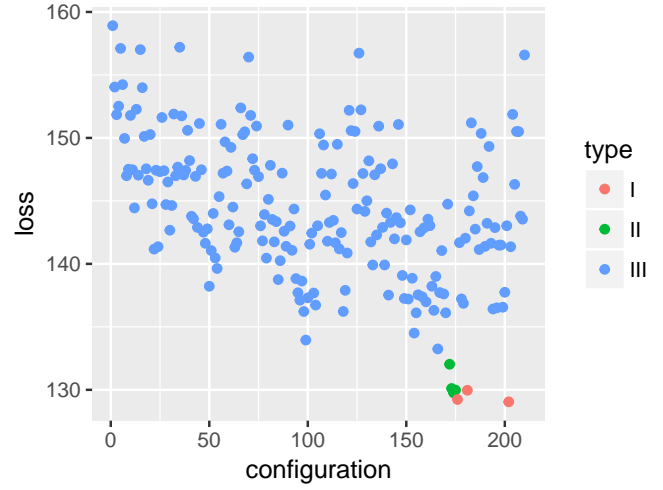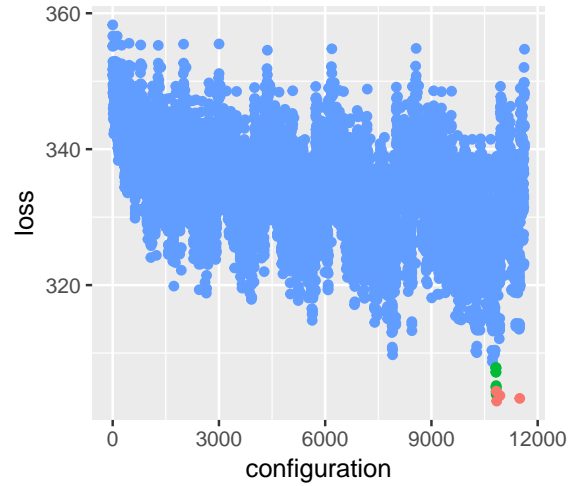
(b) 3-skills case, slip=guess=0.2, J=4

(c) 3-skills case, slip=guess=0.01, J=8

(d) 3-skills case, slip=guess=0.3, J=8

(e) 4-skills case, slip=guess=0.01, J=5

(f) 4-skills case, slip=guess=0.2, J=5

Figure 3: Experiment 2: Configurations of different slip and guess parameters and number of skills, J.

the main finding is wide reaching and warrants further investigations. The support for designing Q-matrices that satisfy the identifiability condition by single-skill items is compelling in the experiments conducted with synthetic data. The results clearly show such matrices yield more accurate student skills assessment. In particular, they show that Q-matrices that contains items that span the whole range of potential combinations of skills tend to yield lower skills assessment than Q-matrices that simply repeat the pattern of single-skill items.

The finding that tests composed of single-skill items are better for skills assessment is somewhat counter-intuitive, as intuition suggests that a good test should also include items with combinations of skills. But intuition also suggests that items that involve combination of skills are more difficult, and it may not simply be because they involve more than one skill. It might be that solving items that combine different skills in a single problem is a new skill in itself. This conjecture is in fact probably familiar to a majority of educators, and the current work provides formal evidence to support it. And the immediate consequence is that Q-matrices, as we currently conceive them, fail to reflect that a task that combines skill involves a new skill.

Ideally, future work should be conducted with real data. However, given that we do not know the real Q-matrix that underlies real data, investigating the questions raised by the current study is non trivial. Meanwhile, further experiments with synthetic data can be considered with different choices on student profiles distribution, and different number of skills involved. Besides, the case where slip and guess are unknown should also be considered, which involves a different identifiability requirement (G. Xu & Zhang, 2015).

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